

Fermions and the Sch/nrCFT Correspondence

Robert G. Leigh and Nam Nguyen Hoang

Department of Physics, University of Illinois, 1110 W. Green St., Urbana IL 61801-3080, U.S.A.

Abstract

We consider the problem of Dirac fermions propagating on a spacetime of Schrödinger isometry and the associated boundary Euclidean two-point function of fermionic scaling operators of the holographic dual non-relativistic conformal theory. Paying careful attention to the representations of the Schrödinger algebra that appear in this problem, we show carefully how the on-shell action is constructed and how the boundary theory may be renormalized consistently by the inclusion of appropriate Galilean invariant boundary counterterms.

1 Introduction

The gauge/gravity holographic correspondence has in recent years been applied to an ever widening array of interesting (presumably strongly coupled) quantum field theories. It is believed that the holographic dictionary can be applied to cases in which the asymptotic geometry of the dual spacetime has isometry that differs from that of AdS. Since this asymptotic symmetry is reflected in the space-time symmetry of the field theory, one can obtain non-Poincaré invariant theories. An interesting example was introduced by Son [1], in which the isometry is the conformal Schrödinger symmetry. This geometrizes both dilatations as well as a phase symmetry that is related to 'particle number'. An extension to finite temperature and density was described in [2, 3, 4, 5].

Although the problem of finite density is of most direct physical interest, the system poses some interesting problems even in the 'vacuum' geometry that possesses the full Schrödinger isometry. We believe that these issues should be sorted out before the finite temperature and density cases can be fully appreciated, and in fact it is possible to do so with complete precision as exact analytic solutions exist, as they do in the relativistic case.

The Schrödinger conformal group has been of interest for many years as the full symmetry of the free Schrödinger wave equation (as well as a few systems with certain interactions). The reader will find discussions in the literature of the representation theory in many papers, particularly in low spatial dimensional cases [6, 7, 8, 9].

The Euclidean correlation functions of scalar operators have been investigated by Refs. [1, 10, 11] and a variety of Lorentzian correlators were discussed extending the methods of Skenderis and van Rees [12, 13] in [14]. In the latter paper, we showed in particular that the boundary renormalization program can be carried out for scalars. The boundary renormalization has not, to our knowledge, been completely sorted out particularly with respect to the metric, as the Fefferman-Graham expansion that governs the asymptotically AdS spacetimes is not available, and must be replaced by a more intricate structure.

In the present paper, we investigate fermionic operators in the vacuum $z = 2$ Schrödinger geometry. The system is significantly different from the corresponding relativistic case, as might be expected from the above comments as well as the somewhat more rich representation theory. Nevertheless, we are able to show that a sensible Dirichlet problem exists for fermions. As in the relativistic case, the bulk on-shell action vanishes, and the on-shell action is determined entirely by boundary terms. These boundary terms are uniquely determined by the requirements of a sensible Dirichlet canonical structure and finiteness. In particular, we explain the structure of possible boundary terms (which are required to preserve the Galilean symmetry of the regulated boundary theory) and show that the on-shell action is finite with the inclusion of a finite number of local boundary counterterms.

2 Background

In standard coordinates, the metric of Schrödinger isometry may be written

$$ds^2 = \frac{L^2}{z^2} \left(-\frac{\beta^2}{z^2} dt^2 + 2dt d\xi + d\vec{x}^2 + dz^2 \right) \quad (1)$$

The \vec{x} are coordinates in d -dimensional space. Although we are primarily interested in Euclidean correlator, we will be working on the Lorentzian geometry, as the former can be easily obtained by a Wick rotation. Rather than giving a conformal class as in the relativistic (AdS) case, the quantity in parentheses is a metric of the Bargman type, in which the coordinate ξ is null. The Killing field $N = \partial_\xi$ generates the central extension of the Schrödinger algebra whose eigenvalue would be interpreted as ‘mass’ or ‘particle number’ in a weakly coupled non-relativistic particle theory. In the present context, fields propagating in the bulk are to be taken to be equivariant with respect to N

$$N\Psi = in\psi \quad (2)$$

and dual quasi-primary operators are labeled by both conformal dimension and n . The ξ -direction is taken to be compact so that the spectrum of dual operators is discrete.

A convenient basis of orthonormalized ($\langle e^a, e^b \rangle = \eta^{ab}$) one-forms is

$$e^0 = \frac{L}{z} \left[\frac{z}{\beta} d\xi - \frac{\beta}{z} dt \right], \quad e^v = \frac{L}{\beta} d\xi, \quad e^r = \frac{L}{z} dz, \quad e^i = \frac{L}{z} dx^i \quad (3)$$

which are dual to the orthonormal basis vectors

$$e_0 = -\frac{z}{L} \frac{z}{\beta} \partial_t, \quad e_v = \frac{z}{L} \left[\frac{z}{\beta} \partial_t + \frac{\beta}{z} \partial_\xi \right], \quad e_r = \frac{z}{L} \partial_z, \quad e_i = \frac{z}{L} \partial_i \quad (4)$$

The Levi-Civita connection has non-zero components

$$\omega^0_r = \frac{1}{L} (e^v - 2e^0), \quad \omega^0_v = -\frac{1}{L} e^r, \quad \omega^v_r = -\frac{1}{L} e^0, \quad \omega^i_r = -\frac{1}{L} e^i \quad (5)$$

Correspondingly, the non-zero components of the Christoffel symbols are

$$\Gamma^z_{zz} = -\frac{1}{z} = \Gamma^{\xi}_{z\xi} = \Gamma^t_{zt} \quad (6)$$

$$\Gamma^i_{zj} = -\frac{1}{z} \delta^i_j, \quad \Gamma^z_{ij} = \frac{1}{z} \delta_{ij} \quad (7)$$

$$\Gamma^z_{t\xi} = \frac{1}{z}, \quad \Gamma^z_{tt} = -2\frac{\beta^2}{z^3}, \quad \Gamma^{\xi}_{zt} = \frac{\beta^2}{z^3} \quad (8)$$

2.1 Dirac Operator

The spin connection in the spinor representation is obtained by writing $\omega^a_b = \omega^A(T^A)^a_b$ and replacing the generators by those in the spinor representation. Since the local group is $SO(d+2, 1)$,

the index A can be thought of as an antisymmetric pair of vector indices. We will use a basis for the Clifford algebra

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \quad (9)$$

where $a, b, \dots = r, 0, v, i$. It is always possible to take a basis in which $\gamma^v \gamma^0$ and γ^r are *Hermitian*.¹ For example, a convenient basis is of the form

$$\gamma^0 = -i\sigma_2 \otimes 1, \quad \gamma^v = -\sigma_1 \otimes 1, \quad \gamma^i = \sigma_3 \otimes \tau^i, \quad \gamma^r = \sigma_3 \otimes \tau^r \quad (10)$$

where τ^i are a representation of $Cl(d)$. We have $(T^{[ab]})^\alpha_\beta = -\frac{i}{4}([\gamma^a, \gamma^b])^\alpha_\beta$. Thus, the spin connection takes the form

$$\omega^\alpha_\beta \sim \omega^a_b([\gamma_a, \gamma^b])^\alpha_\beta. \quad (11)$$

The γ 's are numerical matrices and their indices are those of the local frame, raised and lowered with η . The Dirac operator in general may be written

$$\mathcal{D} = \gamma^c \left(e_c + \frac{1}{4} e_c^\mu \omega_\mu^a{}_b (\gamma_a \gamma^b) \right). \quad (12)$$

It will be convenient to split off the radial part

$$\mathcal{D} = z [\gamma^r \partial_z + \gamma^i \partial_i] + \frac{z^2}{\beta} (\gamma^v - \gamma^0) \partial_t + \beta \gamma^v \partial_\xi - \frac{1}{2} \gamma^r [(d+2)1 - \gamma_v \gamma_0]. \quad (13)$$

The problem can be organized by defining projection operators

$$P_\pm = \frac{1 \pm \gamma^r}{2}, \quad Q_\pm = \frac{1 \pm \gamma^v \gamma^0}{2} \quad (14)$$

These commute with each other, so we can simultaneously diagonalize γ^r and $\gamma^v \gamma^0$. We note that these projection operators appear here naturally because *they commute with $spin(d) \subset spin(d+2, 1)$* . Thus, we can be sure that full $spin(d)$ representations occur for each of the four projection sectors.

Noting then that we can rewrite

$$\gamma^v - \gamma^0 = -2Q_+ \gamma^0 = -2\gamma^0 Q_-, \quad \gamma^v = (Q_- - Q_+) \gamma^0 = \gamma^0 (Q_+ - Q_-) \quad (15)$$

the Dirac operator becomes

$$\mathcal{D} = z [(P_+ - P_-) \partial_z + \gamma^i \partial_i] - 2 \frac{z^2}{\beta} \gamma^0 Q_- \partial_t + \beta \gamma^0 (Q_+ - Q_-) \partial_\xi - \frac{1}{2} (P_+ - P_-) [(d+2)1 + (Q_+ - Q_-)] \quad (16)$$

Before proceeding with the solution of the Dirac equation, we will consider some details of the bulk representation of the Schrödinger algebra that will be important in interpreting the solutions.

¹In particular, to be definite, we will take γ^0 to be anti-Hermitian and γ^v to be Hermitian.

2.2 The Schrödinger Algebra

In a local frame in the bulk, a field will carry a representation of $spin(d+2,1)$. Globally the geometry has isometries given by the Killing vectors

$$N = \partial_\xi \quad (17)$$

$$D = 2t\partial_t + \vec{x} \cdot \vec{\partial} + z\partial_z \quad (18)$$

$$H = \partial_t \quad (19)$$

$$C = t^2\partial_t + t\vec{x} \cdot \vec{\partial} - \frac{1}{2}\vec{x}^2\partial_\xi + tz\partial_z - \frac{1}{2}z^2\partial_\xi \quad (20)$$

$$M_{ij} = x_i\partial_j - x_j\partial_i \quad (21)$$

$$\vec{K} = -t\vec{\partial} + \vec{x}\partial_\xi \quad (22)$$

$$\vec{P} = \vec{\partial} \quad (23)$$

where we have written explicitly the bulk representation on functions. We note that M_{ij} generate $spin(d)$ and that $\{D, H, C\}$ together generate $sl(2, \mathbb{R})$. Since $sl(2, \mathbb{R}) \sim so(2, 1)$, we recognize $spin(d) \oplus sl(2, \mathbb{R}) \subset spin(d+2, 1)$. Thus the representation theory can be understood quite simply in terms of highest weight modules. It is traditional to take D diagonal with eigenvalues referred to generically as Δ , and since N is central, it can be diagonalized as well, and in fact there are super-selection sectors labelled by the eigenvalue of N

$$\Psi(z, t, \xi, \vec{x}) = e^{in\xi} \Psi(z, t, \vec{x}). \quad (24)$$

We refer to such functions as being equivariant.² Highest weight states of fixed n, Δ correspond directly to quasi-primary operators $\Psi(t=0, \vec{x}=0)$ in the boundary theory (at $z=0$).³

However, for a non-trivial representation, we expect that the generators are modified accordingly, and this structure is not complete. For example, we know (and we will verify below) that the $spin(d)$ generator should be replaced by

$$M_{ij} = x_i\partial_j - x_j\partial_i + \Sigma_{ij} \quad (25)$$

when acting on spinor fields. To see how this comes about, we can consider doing Schrödinger transformations on the bulk spinor field. The spinor will transform by the Lie derivative[18]

$$\Psi \rightarrow \mathcal{L}_\xi \Psi = \nabla_\xi \Psi + \frac{1}{8} \langle \nabla_{e_a} \xi, e_b \rangle [\gamma^a, \gamma^b] \Psi \quad (26)$$

In this way, \mathcal{L}_ξ commutes with the Dirac operator, at least as long as ξ is a Killing vector.

²According to [15], the bulk spacetime should be thought of as the total space of a fibre bundle over non-relativistic space-time, with ξ the fibre coordinate. In this sense, eq. (24) is interpreted as meaning that the field is a section of the associated bundle of charge n . See also Refs.[16, 17].

³Here, by boundary we will simply mean the limit $z \rightarrow 0$. Further discussion may be found in a later section.

Let us warm up by evaluating the Lie derivative for the $spin(d)$ Killing vector

$$\xi_M = 2x_j \Theta^{ji} \partial_i \quad (27)$$

where Θ is antisymmetric. In this case, we compute

$$\nabla_{\xi_M} \Psi = 2x_j \Theta^{ji} \partial_i \Psi - \frac{1}{z} x_j \Theta^{ji} \gamma_i \gamma^r \Psi \quad (28)$$

$$\langle \nabla_{e_a} \xi_M, e_b \rangle [\gamma^a, \gamma^b] \Psi = 4\Theta_{ji} \left[\gamma^j \gamma^i + 2\frac{1}{z} x^j \gamma^i \gamma^r \right] \Psi \quad (29)$$

This result does indeed correspond to (25). Now for ξ_K , we have

$$\xi_K = -t\vec{K} \cdot \vec{\partial} + (\vec{x} \cdot \vec{K}) \partial_\xi \quad (30)$$

and we find

$$\mathcal{L}_K \Psi = (\vec{x} \cdot \vec{K}) \partial_\xi \Psi - t\vec{K} \cdot \vec{\partial} \Psi + \frac{1}{2\beta} z (\gamma^0 - \gamma^v) K_i \gamma^i \Psi \quad (31)$$

By similar computations, we conclude that the representation on spinors is

$$M_{ij} = x_i \partial_j - x_j \partial_i + \frac{1}{4} [\gamma_i, \gamma_j] \equiv x_i \partial_j - x_j \partial_i + i\Sigma_{ij} \quad (32)$$

$$K_i = -t\partial_i + x_i \partial_\xi + \frac{1}{2} \frac{z}{\beta} (\gamma^0 - \gamma^v) \gamma_i \equiv -t\partial_i + x_i \partial_\xi + \kappa_i \quad (33)$$

$$C = tz\partial_z + t\vec{x} \cdot \vec{\partial} + t^2 \partial_t - \frac{1}{2} (\vec{x}^2 + z^2) \partial_\xi + \hat{c} \quad (34)$$

where $\hat{c} = +\frac{1}{2} \frac{z}{\beta} (\not{x} + z\gamma^r) (\gamma^0 - \gamma^v)$. Notice that both κ_i and \hat{c} are nilpotent. D, H, P_i are unmodified. The highest weight states of a Schrödinger multiplet are annihilated by both C and K_i , and so we see here that these conditions apparently do not act diagonally on $spin(d)$ components of the bulk Dirac spinor, but mix them at a generic point in the bulk. However, as we will see, the on-shell Dirac spinor is constructed from chiral spinors that are Q -chiral and as we show in detail in the appendix, \hat{c} and κ are such that K_i and C act diagonally on these.

3 Solutions of the Dirac Equation

Now we write spinors as linear combinations of doubly chiral spinors

$$\Psi(z, t, \xi, \vec{x}) = e^{in\xi} \sum_{\varepsilon_r, \varepsilon_\ell} \int \frac{d\omega}{2\pi} \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k} \cdot \vec{x}} e^{-i\omega t} \psi_{n, \omega, \vec{k}}^{\varepsilon_r, \varepsilon_\ell}(z) \quad (35)$$

where

$$\gamma^r \psi_{n, \omega, \vec{k}}^{\varepsilon_r, \varepsilon_\ell}(z) = \varepsilon_r \psi_{n, \omega, \vec{k}}^{\varepsilon_r, \varepsilon_\ell}(z) \quad (36)$$

$$\gamma^v \gamma^0 \psi_{n, \omega, \vec{k}}^{\varepsilon_r, \varepsilon_\ell}(z) = \varepsilon_\ell \psi_{n, \omega, \vec{k}}^{\varepsilon_r, \varepsilon_\ell}(z) \quad (37)$$

and $\varepsilon_r, \varepsilon_l = \pm 1$. By projecting the Dirac equation with P_\pm and Q_\pm , we find four equations (we drop the subscripts on ψ for brevity)

$$\left(z\partial_z - \frac{1}{2}(d+3) + m_0\right) \gamma^0 \psi^{-,+} + iz\rlap{\not{\mathbb{K}}}\gamma^0 \psi^{+,+} + \frac{i}{\beta}(2\omega z^2 - n\beta^2) \psi^{+,-} = 0 \quad (38)$$

$$\left(z\partial_z - \frac{1}{2}(d+1) - m_0\right) \psi^{+,-} + iz\rlap{\not{\mathbb{K}}}\psi^{-,-} + in\beta\gamma^0 \psi^{-,+} = 0 \quad (39)$$

$$\left(z\partial_z - \frac{1}{2}(d+3) - m_0\right) \gamma^0 \psi^{+,+} - iz\rlap{\not{\mathbb{K}}}\gamma^0 \psi^{-,+} - \frac{i}{\beta}(2\omega z^2 - n\beta^2) \psi^{-,-} = 0 \quad (40)$$

$$\left(z\partial_z - \frac{1}{2}(d+1) + m_0\right) \psi^{-,-} - iz\rlap{\not{\mathbb{K}}}\psi^{+,-} - in\beta\gamma^0 \psi^{+,+} = 0 \quad (41)$$

Note that we have used the notation $\rlap{\not{\mathbb{K}}}$ to emphasize that this is the quantity in the trivial metric. The invariant is $\rlap{\not{\mathbb{K}}} = \gamma^a e_a^i k_i$, and in the Schrödinger metric, this evaluates to $\rlap{\not{\mathbb{K}}} = \frac{z}{L}\rlap{\not{\mathbb{K}}}$. This accounts for the single powers of z accompanying $\rlap{\not{\mathbb{K}}}$; this will be of additional importance later in the context of boundary renormalization.

It is not difficult to disentangle the Dirac equations, and we find in particular that

$$\left[z^2\partial_z^2 - (d+1)z\partial_z + \left(\frac{d}{2} + 1\right)^2 - \mu_{\varepsilon_r}^2 - q^2 z^2\right] \psi^{\varepsilon_r,-} = 0 \quad (42)$$

where

$$\mu_{\varepsilon_r} = \sqrt{\left(\frac{1}{2} - \varepsilon_r m_0\right)^2 + n^2 \beta^2} \quad (43)$$

and

$$q^2 = \vec{k}^2 - 2n\omega \quad (44)$$

In this paper, we consider only the case $n \neq 0$. We note that q^2 appears naturally, as it is the Fourier transform of the Galilean-invariant Schrödinger operator $\mathcal{S} = i\partial_t - \frac{1}{2n}\vec{\nabla}^2$. In particular, we note that when acting on equivariants

$$[K_i, \mathcal{S}] = 0 \quad (45)$$

This fact will play a central role later in our discussion of renormalizability.

Thus we have

$$\psi^{\varepsilon_r,-}(\vec{k}, \omega, z) = z^{1+d/2} K_{\mu_{\varepsilon_r}}(qz) \mathbf{u}_{\varepsilon_r}(\vec{k}, \omega) \quad (46)$$

where \mathbf{u}_\pm are independent doubly chiral $spin(d)$ spinors that satisfy

$$Q_+ \mathbf{u}_\pm = 0, \quad \gamma^r \mathbf{u}_\pm = \pm \mathbf{u}_\pm \quad (47)$$

Since we are interested in Euclidean correlator, we have dropped the solution proportional to $I_{\mu_{\varepsilon_r}}(qz)$ by requiring regularity at large z . Substituting these solutions back into the Dirac equations leads algebraically to the other components of the Dirac spinor

$$\psi^{\pm,+} = \pm \frac{i}{n\beta} z^{1+d/2} \gamma^0 \left[\left(qz K'_{\mu_\mp}(qz) + \left(\frac{1}{2} \pm m_0\right) K_{\mu_\mp}(qz) \right) \mathbf{u}_\mp(\vec{k}, \omega) \mp iz K_{\mu_\pm}(qz) \rlap{\not{\mathbb{K}}} \mathbf{u}_\pm(\vec{k}, \omega) \right]$$

The general on-shell field then is

$$\Psi = -z^{1+d/2} \sum_{\varepsilon_r} \left[\frac{i\varepsilon_r}{n\beta} \left(\frac{1}{2} - \varepsilon_r m_0 - i\varepsilon_r n\beta \gamma^0 \right) K_{\mu_{\varepsilon_r}}(qz) + \frac{i\varepsilon_r}{n\beta} qz K'_{\mu_{\varepsilon_r}}(qz) + \frac{z}{n\beta} K_{\mu_{\varepsilon_r}}(qz) \not{\mathbb{I}} \right] \gamma^0 \mathbf{u}_{\varepsilon_r} \quad (48)$$

From this general solution, we see that the leading terms at the boundary are

$$\Psi \sim \sum_{\varepsilon_r} z^{\Delta_{\varepsilon_r}^-} \frac{\Gamma(\mu_{\varepsilon_r})}{2} \left(\frac{q}{2} \right)^{-\mu_{\varepsilon_r}} X_{-(0)}^{\varepsilon_r} \mathbf{u}_{\varepsilon_r} + \dots \quad (49)$$

where $X_{-(0)}^{\varepsilon_r} \equiv 1 - \frac{i\varepsilon_r}{n\beta} \left(\frac{1}{2} - \varepsilon_r m_0 - \mu_{\varepsilon_r} \right) \gamma^0$ and where

$$\Delta_{\varepsilon_r}^{\pm} = 1 + \frac{d}{2} \pm \mu_{\varepsilon_r}, \quad (50)$$

We make several comments:

- For generic bulk mass, the two dimensions $\Delta_{\varepsilon_r}^-$ are irrationally related. Thus \mathbf{u}_+ and \mathbf{u}_- must be taken as independent sources. The only counter-example (for $n \neq 0$) is if the bulk mass vanishes, in which case $\mu_+ = \mu_-$ and $\Delta_+^- = \Delta_-^- \equiv \Delta^-$. Thus the massless case is special, in that the eigenvalues of D are degenerate. One can show however that in either case, \mathbf{u}_{\pm} transform separately under the Schrödinger algebra.
- More precisely, the coefficient of the leading singularity is of the form $X_{-(0)}^{\varepsilon_r} \mathbf{u}_{\varepsilon_r}$ and is not chiral. We note though that the $X_{-(0)}^{\varepsilon_r}$ are *constant* matrices and can be absorbed (by a basis change) into the definition of boundary operators.
- The dependence on $\not{\mathbb{I}}$ is subleading in the near-boundary ($z \rightarrow 0$) expansion, and is associated with the *odd* powers of z .
- The general solution is obtained by specifying two $spin(d)$ spinors \mathbf{u}_{\pm} that are Q -chiral ($Q_+ \mathbf{u}_{\pm} = 0$). Thus the Dirac equation has eliminated half of the degrees of freedom, as expected.
- The Schrödinger covariance of this expression is not manifest as written. To see this, consider the symplectic 1-form

$$\alpha = -idx^{\mu} \otimes \partial_{\mu} \quad (51)$$

which when acting on plane waves gives

$$\alpha = -idz \otimes \partial_z + (nd\xi - \omega dt + \vec{k} \cdot d\vec{x}) \quad (52)$$

The scalar Lagrangian, for example, can be written

$$S_{scalar} \sim \int \sqrt{-g} \langle \alpha(\phi), \alpha(\phi) \rangle = - \int \sqrt{-g} \phi^{\dagger} \Delta \phi \quad (53)$$

For fermions, we replace the exterior algebra by the Clifford algebra, and hence we obtain

$$\alpha \rightarrow -i\frac{z}{L}\gamma^r\partial_z + \frac{n\beta}{L}\gamma^v + \omega\frac{z^2}{L\beta}(\gamma^0 - \gamma^v) + \frac{z}{L}\not{\omega} \quad (54)$$

This of course is the quantity appearing in the Dirac operator, apart from the spin connection terms. What we learn from this is that the $\not{\omega}$ term should really be grouped with

$$\mathcal{Q} = z\not{\omega} + n\beta\gamma^v + 2\omega\frac{z^2}{\beta}\gamma^0\mathcal{Q}_- \quad (55)$$

In particular, this combination is Galilean invariant, $[K_i, \mathcal{Q}] = 0$, and also $[D, \mathcal{Q}] = 0$. (Later, it will play a central role in the renormalizability of the theory.) We also note that

$$\mathcal{Q}^2 = z^2\mathcal{S} + n^2\beta^2 \quad (56)$$

which is the scalar invariant noted above, where $\mathcal{S} = q^2 = \vec{k}^2 - 2n\omega$.

Furthermore, when acting on the chiral spinors $\gamma^0\mathbf{u}_\pm$, this simplifies to

$$\mathcal{Q}\gamma^0\mathbf{u}_\pm = n\beta\left(\gamma^0 + \frac{z\not{\omega}}{n\beta}\right)\gamma^0\mathbf{u}_\pm \quad (57)$$

and thus the on-shell spinor can be rewritten

$$\Psi = -\frac{i}{n\beta}z^{1+d/2}\sum_{\varepsilon_r}\left[\varepsilon_rqzK'_{\mu_{\varepsilon_r}}(qz) - (m_0 - \varepsilon_r/2)K_{\mu_{\varepsilon_r}}(qz) - iK_{\mu_{\varepsilon_r}}(qz)\mathcal{Q}\right]\gamma^0\mathbf{u}_{\varepsilon_r} \quad (58)$$

We note that $\mathbf{v}_{\varepsilon_r} \equiv \gamma^0\mathbf{u}_{\varepsilon_r}$ are chiral, with $\mathcal{Q}_-\mathbf{v}_{\varepsilon_r} = 0$. We note though that the individual pieces of \mathcal{Q} come in with different powers of z , and so going to the boundary is somewhat subtle. What we must do is understand how the generators of the Schrödinger algebra act on the terms in the expansion of the field. This is explained in detail in the Appendix.

4 Variational Principle and Boundary Renormalization

As in the relativistic case [19, 20, 21, 22], we need to add a boundary term to the Dirac action, as the bulk part of the Dirac action vanishes on-shell. This boundary term serves to give the proper Dirichlet boundary condition and simultaneously make the on-shell action finite. Since in the variational principle the field variations are off-shell, first of all we have to state clearly what we mean by an off-shell spinor. The on-shell solution (58) suggests that an off-shell spinor should have the same z expansion, except that the coefficients are in general full unconstrained Dirac spinors. This off-shell spinor obviously carries a Schrodinger representation through (32)-(34). However, as established in the Appendix, this representation is reducible. Hence, it is natural that we define the off-shell spinor to be an irreducible representation that encompasses all vacuum solutions (58). According to (A.19), it takes the form

$$\Psi = \sum_{\varepsilon_r}\left[z^{\Delta_{-\varepsilon_r}^-}\sum_{k=0}^{\infty}z^{2k}\left(\rho_{(2k)}^{\varepsilon_r} + \mathcal{Q}\rho_{(2k+1)}^{\varepsilon_r}\right) + z^{\Delta_{-\varepsilon_r}^+}\sum_{k=0}^{\infty}z^{2k}\left(\chi_{(2k)}^{\varepsilon_r} + \mathcal{Q}\chi_{(2k+1)}^{\varepsilon_r}\right)\right] \quad (59)$$

where \mathcal{Q} is given by (55) and

$$\gamma^r \rho_{(m)}^{\varepsilon_r} = \varepsilon_r \rho_{(m)}^{\varepsilon_r}, \quad Q_- \rho_{(m)}^{\varepsilon_r} = 0 \quad (60)$$

similarly for χ .

By comparison to the on-shell solution, one can deduce the on-shell relationship of $\rho_{(m)}^{\varepsilon_r}$ and $\chi_{(m)}^{\varepsilon_r}$ to the independent quantities $\rho_{(0)}^{\varepsilon_r} \sim \mathbf{u}_{\varepsilon_r}$

$$\rho_{(2k)}^{\varepsilon_r} = -i x_{-(2k)}^{\varepsilon_r} \rho_{(2k+1)}^{\varepsilon_r} = \frac{x_{-(2k)}^{\varepsilon_r}}{x_{-(0)}^{\varepsilon_r}} \frac{\Gamma(1 - \mu_{-\varepsilon_r})}{k! \Gamma(k + 1 - \mu_{-\varepsilon_r})} \left(\frac{q}{2}\right)^{2k} \rho_{(0)}^{\varepsilon_r} \quad (61)$$

$$\chi_{(2k)}^{\varepsilon_r} = -i x_{+(2k)}^{\varepsilon_r} \chi_{(2k+1)}^{\varepsilon_r} = -\frac{x_{+(2k)}^{\varepsilon_r}}{x_{-(0)}^{\varepsilon_r}} \frac{\Gamma(1 - \mu_{-\varepsilon_r})}{k! \Gamma(k + 1 + \mu_{-\varepsilon_r})} \left(\frac{q}{2}\right)^{2k+2\mu_{-\varepsilon_r}} \rho_{(0)}^{\varepsilon_r}, \quad (62)$$

where $x_{\pm(2k)}^{\varepsilon_r} = \varepsilon_r (\frac{1}{2} + \varepsilon_r m_0 \pm \mu_{-\varepsilon_r} + 2k)$.

As is well known, the bulk part of the Dirac action evaluates to zero on the equations of motion. The on-shell action is determined entirely by the boundary term. The renormalized Dirac action must be of the form

$$S_{Lor} = \int_M d^{d+3}x \sqrt{-g} \bar{\Psi} i(\not{D} - m_0) \Psi + \frac{1}{L^2} \int_{\partial M} dt d\xi d^d x \sqrt{\gamma} e_r^z \bar{\Psi} T \Psi \quad (63)$$

for some matrix T , which must respect the symmetries of the boundary theory. Given that

$$\Psi(z, t, \xi, \vec{x}) = e^{in\xi} \Psi(z, t, \vec{x}) \quad (64)$$

the action reduces to

$$R \int d^{d+1}x \int dz \sqrt{-g} \bar{\Psi} i(\not{D}_n - m_0) \Psi + R \int dt d^d x \sqrt{\gamma} z^{-d-2} \bar{\Psi} T \Psi \quad (65)$$

where γ denotes the spatial induced d -metric of the boundary (which in our case is flat). We absorb this overall factor of R into the normalization. We vary the action subject to the vanishing of the variation of the source. Given our choice of action, we find

$$\delta S = \int \bar{\Psi} (i\gamma^r + T) \delta \Psi + \int \delta \bar{\Psi} T \Psi \quad (66)$$

A proper variational principle is obtained by requiring that terms involving $\delta\chi$ not appear in this expression. This will force the variational principle to give the correct Dirichlet condition $\delta\rho = 0$ on the boundary. In addition to this requirement, the resulting on-shell action must be made finite by the addition of suitable boundary counter-terms. As is customary, we will use minimal subtraction. By suitable, we mean any term that respects the symmetry of the regulated boundary theory. In the case of AdS, the boundary counterterms are Poincaré invariant, which is the symmetry respected by the regulator. In our case, we expect the counterterms to be Galilean invariant. Since these counterterms are all written in terms of the boundary values of bulk fields,

upon which the Schrödinger transformations act in the prescribed way, it is appropriate to write the boundary counterterms as boundary values of bulk-invariant terms (that is, using the bulk metric for contractions and the bulk realization of the symmetry generators).

Requiring the boundary term to be Galilean invariant, T has to be written in terms of operators that commute with the Galilean generators, in particular the K_i . Careful consideration of this problem reveals that such invariants may be constructed out of P_{ε_r} , $\gamma^0 Q_-$ and \mathcal{Q} and thus the most general boundary term can be written as (here $L = \mathcal{Q}^2/n^2\beta^2$)⁴

$$T = \sum_{\varepsilon_r} \left[a_{\varepsilon_r}(L) + b_{\varepsilon_r}(L) \frac{i\mathcal{Q}}{n\beta} + \left[c_{\varepsilon_r}(L) + d_{\varepsilon_r}(L) \frac{i\mathcal{Q}}{n\beta} \right] \gamma^0 Q_- \right] P_{\varepsilon_r} \quad (67)$$

where $a_{\varepsilon_r}(L), \dots$ are functions to be determined. Although we have written the coefficient functions as functions of $\mathcal{Q}^2 \sim L$ for notational brevity, since N is central and the fields are equivariant, this could just as well be replaced by $z^2 \mathcal{S} = z^2 q^2$.

This form for T and the field written in the form (58) is most convenient to discuss the renormalizability of the theory – it organizes the counterterms in an invariant fashion. What is more complicated here, compared to the relativistic case, is that this organization is not homogeneous in powers of z . It is easy to see in this form however how the renormalization will work – since $\mathcal{Q}^2 \sim L$, we can regard T as an expansion in powers of \mathcal{Q} (rather than z). At any given order, canceling divergences will correspond to conditions on the Taylor coefficients of the functions $a_{\varepsilon_r}(L), \dots$ around $L = 1$. Depending on the values of various parameters (m_o, n, \dots), we can terminate the Taylor expansion at some order, as all further contributions to the action will be zero when the cutoff is removed. It remains then to demonstrate that the conditions on the Taylor coefficients can be consistently solved to remove all divergences. We will not construct a general proof, and in fact will work just at lowest order. Experience with these manipulations suggests that no problems will be encountered at higher orders.

Given the form for T , we consider the variational problem; this will place conditions on the lowest order Taylor coefficients of the functions in T . We write (66) in terms of $\delta\rho_{(m)}^{\varepsilon_r}$ and $\delta\chi_{(m)}^{\varepsilon_r}$ and their conjugates. Due to the fact that $\mu_- - \mu_+ < 1$ for $n > 0$, the only terms that possibly contain $\delta\chi_{(m)}^{\varepsilon_r}$ are the finite term (z^0 power), which evaluate to

$$(\rho_{(0)}^{\varepsilon_r})^\dagger \delta\chi_{(0)}^{\varepsilon_r} \left[-ib_{\varepsilon_r} - \frac{in\beta}{x_{-(0)}^{\varepsilon_r}} (a_{\varepsilon_r} + i\varepsilon_r) \right] + n\beta (\rho_{(0)}^{\varepsilon_r})^\dagger \delta\chi_{(1)}^{\varepsilon_r} \left[i\varepsilon_r - (a_{-\varepsilon_r} - id_{-\varepsilon_r}) + \frac{n\beta}{x_{-(0)}^{\varepsilon_r}} (b_{-\varepsilon_r} + ic_{-\varepsilon_r}) \right] \quad (68)$$

where $a_{\varepsilon_r} \equiv a_{\varepsilon_r}(1)$, etc. To obtain this expression, we have used the on-shell relations for the $\rho_{(m)}^{\varepsilon_r}$. Similarly, the only terms involving $\delta\chi^\dagger$ are

$$-(\delta\chi_{(0)}^{\varepsilon_r})^\dagger \rho_{(0)}^{\varepsilon_r} \left[ib_{\varepsilon_r} + \frac{in\beta}{x_{-(0)}^{\varepsilon_r}} (a_{-\varepsilon_r} - id_{-\varepsilon_r}) \right] + n\beta (\delta\chi_{(1)}^{\varepsilon_r})^\dagger \rho_{(0)}^{\varepsilon_r} \left[a_{\varepsilon_r} - \frac{n\beta}{x_{-(0)}^{\varepsilon_r}} (b_{-\varepsilon_r} + ic_{-\varepsilon_r}) \right] \quad (69)$$

⁴the similar expression for AdS would be $\sum_{\varepsilon_r} (a_{\varepsilon_r}(q^2) + k b_{\varepsilon_r}(q^2)) P_{\varepsilon_r}$, where P_{ε_r} is the projector along the radial direction.

The variational principle requires each term to vanish separately, which results in three independent equations

$$b_{\varepsilon_r} = -\frac{n\beta}{x_{-(0)}^{\varepsilon_r}}(a_{\varepsilon_r} + i\varepsilon_r) \quad (70)$$

$$b_{\varepsilon_r} = -\frac{n\beta}{x_{-(0)}^{\varepsilon_r}}(a_{-\varepsilon_r} - id_{-\varepsilon_r}) \quad (71)$$

$$a_{\varepsilon_r} = \frac{n\beta}{x_{-(0)}^{\varepsilon_r}}(b_{-\varepsilon_r} + ic_{-\varepsilon_r}) \quad (72)$$

As argued above, given a specific value of $a_{\varepsilon_r} = a_{\varepsilon_r}(1), \dots$ satisfying (70)-(72), the higher order coefficients in the Taylor expansion can be found successively by requiring the cancelation of sub-leading divergences. There are, however, two things that those higher order coefficients cannot control, since they involve subleading powers in z . They are the leading divergence, of the form $\rho_{(0)}^{\varepsilon_r \dagger} \rho_{(0)}^{\varepsilon_r}$ ⁵, and the finite part of the on-shell action. Requiring the leading divergence to be zero sets

$$-ib_{\varepsilon_r} - \frac{in\beta}{x_{-(0)}^{\varepsilon_r}}(a_{-\varepsilon_r} - id_{-\varepsilon_r}) + \frac{in\beta}{x_{-(0)}^{\varepsilon_r}} \left(a_{\varepsilon_r} - \frac{n\beta}{x_{-(0)}^{\varepsilon_r}}(b_{-\varepsilon_r} + ic_{-\varepsilon_r}) \right) = 0, \quad (73)$$

which is fortunately automatically satisfied from (71) and (72). The finite part of the on-shell action is in fact independent of the values of a_{ε_r}, \dots (although the variation of the action is not). Indeed, a short calculation gives

$$\begin{aligned} S_{os} &= \int \frac{d\omega}{2\pi} \int \frac{d^d k}{(2\pi)^d} \sum_{\varepsilon_r} \frac{2\varepsilon_r \mu_{-\varepsilon_r}}{n\beta} (\rho_{(0)}^{\varepsilon_r})^\dagger \chi_{(0)}^{\varepsilon_r} \\ &= -\frac{2\varepsilon_r \mu_{-\varepsilon_r}}{n\beta} \int \frac{d\omega}{2\pi} \int \frac{d^d k}{(2\pi)^d} \sum_{\varepsilon_r} \frac{x_{+(2k)}^{\varepsilon_r}}{x_{-(0)}^{\varepsilon_r}} \frac{\Gamma(1 - \mu_{-\varepsilon_r})}{\Gamma(1 + \mu_{-\varepsilon_r})} \left(\frac{q^2}{4} \right)^{\mu_{-\varepsilon_r}} (\rho_{(0)}^{\varepsilon_r})^\dagger \rho_{(0)}^{\varepsilon_r}, \end{aligned} \quad (74)$$

where we have used the conditions (70)-(72). Thus, it is scheme independent.

5 Boundary Operators

As we have seen, the leading term in the expansion of the field is proportional to $X_{-(0)}^{\varepsilon_r} \rho_{(0)}^{\varepsilon_r}$, where $X_{-(0)}^{\varepsilon_r}$ is a constant matrix and $\rho_{(0)}^{\varepsilon_r}$ is a (doubly) chiral spinor field. It is convenient to take a basis of boundary quasi-primary operators such that $\rho_{(0)}^{\varepsilon_r}$ act as the sources for operators of charge n and dimension $\Delta_{\varepsilon_r}^+$

$$\int dt \int d^d x \sqrt{\gamma} \left[(\rho_{(0)}^{\varepsilon_r})^\dagger(t, \vec{x}) \mathcal{O}_{n, \varepsilon_r}(t, \vec{x}) + h.c. \right] \quad (75)$$

⁵Other possible terms such as $\rho_{(0)}^{\varepsilon_r \dagger} \not{K} \rho_{(0)}^{-\varepsilon_r}$ must be thought of as a piece of $\bar{\rho}_{(0)}^{\varepsilon_r} \gamma^0 \mathcal{Q} \rho_{(0)}^{-\varepsilon_r}$, but as we have discussed, this is not K_i invariant, so will not appear.

This is possible because the $X_{-(0)}^{\varepsilon_r}$ are constant matrices. This coupling preserves the Schrödinger invariance at the boundary, obtained from the bulk transformations, for example

$$v^i K_i : \quad \Psi'(t', \vec{x}', z') = e^{in(\vec{v} \cdot \vec{x} + i\vec{v}^2 t/2)} (1 + v \cdot \kappa) \Psi(t, \vec{x}, z) \quad (76)$$

$$cC : \quad \Psi'(t', \vec{x}', z') = e^{-\frac{inc}{2} \frac{\vec{x}^2 + z^2}{1+ct}} (1 + c\hat{c}) \Psi(t, \vec{x}, z), \quad (77)$$

at $z = 0$. Under, say, a finite C transformation, the coupling (75) transforms as

$$\int dt \int d^d x \sqrt{\gamma} (\rho_{(0)}^{\varepsilon_r})^\dagger(t, \vec{x}) \mathcal{O}_{n, \varepsilon_r}(t, \vec{x}) \rightarrow \int \frac{dt d^d x}{(1+ct)^{d+2}} (1+ct)^{\Delta_{-\varepsilon_r}^+ + \Delta_{-\varepsilon_r}^-} (\rho_{(0)}^{\varepsilon_r})^\dagger(t, \vec{x}) \mathcal{O}_{n, \varepsilon_r}(t, \vec{x}) \quad (78)$$

given the appropriate transformation of boundary quasi-primary operators (see for example Ref. [9]). Thus the coupling is invariant, as $\Delta_{\varepsilon_r}^+ + \Delta_{\varepsilon_r}^- = d + 2$.

Given the form of the on-shell action, we then read off the two-point Euclidean correlator of quasi-primary operators

$$\langle (\mathcal{O}_{n, \varepsilon_r})^\dagger(t, \vec{x}) \mathcal{O}_{n', \varepsilon_r'}(t', \vec{x}') \rangle = -\delta_{\varepsilon_r, \varepsilon_r'} \delta_{n, n'} \int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} e^{-i\omega(t'-t)} e^{i\vec{k} \cdot (\vec{x}' - \vec{x})} \frac{2\varepsilon_r \mu_{-\varepsilon_r}}{n\beta} \frac{x_{+(2k)}^{\varepsilon_r}}{x_{-(0)}^{\varepsilon_r}} \frac{\Gamma(1 - \mu_{-\varepsilon_r})}{\Gamma(1 + \mu_{-\varepsilon_r})} \left(\frac{q^2}{4}\right)^{\mu_{-\varepsilon_r}} \quad (79)$$

By scaling, it is easy to see that this behaves as

$$(t' - t)^{-\Delta_{-\varepsilon_r}^+} f\left(\frac{(\vec{x}' - \vec{x})^2}{(t' - t)}\right),$$

and in fact this is just proportional to the scalar propagator. We note that this correlator preserves chirality, and in particular no γ -matrix structure is present. This is expected of a non-relativistic theory, as there is no essential difference between boson and fermion fields.

This is not to say that other correlation functions do not have more interesting structure. The subleading terms in the asymptotic expansion of the field are sources for descendant operators. Given the form of the generators in the bulk (33,34), we see that Schrödinger transformations mix the descendant fields in an interesting way. The correlation functions of dual operators will of course display a similar structure, and thus these correlation functions can have non-trivial γ -matrix structure.

6 Conclusion

We have investigated carefully the Dirac fermion problem on the spacetime of Schrödinger isometry, which is dual to the vacuum configuration of non-relativistic conformal field theories in d spatial dimensions. The structure of the system is rather intricate, but a sensible Dirichlet problem exists and the boundary theory is renormalizable.

Although the bulk geometry contains a compact null direction (coordinatized by ξ), the metric is of the Bargman type and the usual holographic prescriptions go through more or less unmodified for equivariant operators, with care taken in interpreting the boundary action. The bulk field sources operators of a highest weight module of the Schrödinger algebra and the correct structure of two-point correlation functions is obtained. It would be interesting to extend these computations to higher point functions and to finite density.

We should mention that we are aware of one paper concerning fermions in the Schrödinger geometry [23]. We have not been able to understand the details of the computations in this paper or their results.

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A Appendix

As noted in the text, the realization in the bulk of K_i and C acting on fermions is

$$K_i = -t\partial_i + x_i\partial_\xi + \frac{z}{\beta}\gamma^0\gamma_i Q_- \quad (\text{A.1})$$

$$= K_i^{(0)} + \frac{z}{\beta}\gamma^0\gamma_i Q_- \quad (\text{A.2})$$

$$C = t(z\partial_z + \vec{x}\vec{\partial} + t\partial_t) - \frac{1}{2}(\vec{x}^2 + z^2)\partial_\xi + \frac{z}{\beta}(\not{x} + z\gamma^r)\gamma^0 Q_- \quad (\text{A.3})$$

$$= C^{(0)} - \frac{1}{2}z^2\partial_\xi + \frac{z}{\beta}(\not{x} + z\gamma^r)\gamma^0 Q_- \quad (\text{A.4})$$

A.1 off-shell transformation

Motivated by the general solution of the Dirac equation, the off-shell spinors are assumed to have the near boundary expansion

$$\begin{aligned} \Psi = & z^{\Delta_+^-} \sum_{k=0}^{\infty} z^{2k} (\Psi_{(2k)}^I + z \Psi_{(2k+1)}^I) + z^{\Delta_+^+} \sum_{k=0}^{\infty} z^{2k} (\Psi_{(2k)}^{II} + z \Psi_{(2k+1)}^{II}) \\ & + z^{\Delta_-^-} \sum_{k=0}^{\infty} z^{2k} (\Psi_{(2k)}^{III} + z \Psi_{(2k+1)}^{III}) + z^{\Delta_-^+} \sum_{k=0}^{\infty} z^{2k} (\Psi_{(2k)}^{IV} + z \Psi_{(2k+1)}^{IV}) \quad (\text{A.5}) \end{aligned}$$

containing four power series of z , in which the $\Psi_{(m)}$'s are in general full Dirac spinors. For generic values of d , m_0 and n these series do not talk to each other under Schrödinger transformations. Each of them form a separate representation of the Schrödinger group. As our purpose is to work out the transformation laws, it is sufficient to focus on just one of them, say the one with Δ_+^- . Results for the other series are inferred immediately.

Let's re-parametrize the first series in terms of P and Q chiral spinors as follows

$$\Psi^I = \sum_{k, \varepsilon'_r \varepsilon'_\ell} z^{\Delta_+^- + 2k} \left(\rho_{(2k)}^{\varepsilon'_r \varepsilon'_\ell} + \mathcal{Q} \rho_{(2k+1)}^{\varepsilon'_r \varepsilon'_\ell} \right), \quad (\text{A.6})$$

where $\gamma^r \rho_{(m)}^{\varepsilon_r \varepsilon_\ell} = \varepsilon_r \rho_{(m)}^{\varepsilon_r \varepsilon_\ell}$ and $\gamma^v \gamma^0 \rho_{(m)}^{\varepsilon_r \varepsilon_\ell} = \varepsilon_\ell \rho_{(m)}^{\varepsilon_r \varepsilon_\ell}$.

Our task is to find the transformation laws of $\rho_{(m)}^{\varepsilon_r \varepsilon_\ell}$ under the Schrödinger algebra, restricted to the non-trivial isometries K_i and C . We will then argue that it is possible to consistently reduce the representation by setting half of the fields to zero, leaving the $\rho_{(m)}^{\varepsilon_r +}$ untouched. At this point, the remaining fields $\rho_{(m)}^{++}$ and $\rho_{(m)}^{-+}$ transform independently, so one of them can be further set to zero under the criteria that the leftover representation should include all on-shell solutions. In particular, a suitable irreducible representation corresponds to keeping $\rho_{(m)}^{-+}$ for Ψ^I , Ψ^{II} and $\rho_{(m)}^{++}$ for Ψ^{III} , Ψ^{IV} .

Transformations of $\rho_{(m)}^{\varepsilon_r \varepsilon_\ell}$ are straightforwardly found by acting with K_i and C on (A.6) and reading off the coefficients of different powers of z . Projecting further by $P_{\varepsilon_r} Q_{\varepsilon_\ell}$ we get

$$\not{K} \delta_{K_i} \rho_{(2k+1)}^{\bar{\varepsilon}_r \varepsilon_\ell} = \not{K} K_i^{(0)} \rho_{(2k+1)}^{\bar{\varepsilon}_r \varepsilon_\ell} + \frac{1}{\beta} \gamma^0 \gamma_i \rho_{(2k)}^{\varepsilon_r -} \delta_{\varepsilon_\ell, +} - n \gamma_i \rho_{(2k+1)}^{\bar{\varepsilon}_r -} \delta_{\varepsilon_\ell, -} \quad (\text{A.7})$$

$$\not{K} \delta_C \rho_{(2k+1)}^{\bar{\varepsilon}_r \varepsilon_\ell} = C^{(0)} \not{K} \rho_{(2k+1)}^{\bar{\varepsilon}_r \varepsilon_\ell} + \frac{\not{K}}{\beta} \gamma^0 \rho_{(2k)}^{\varepsilon_r -} \delta_{\varepsilon_\ell, +} - \frac{in}{2} \not{K} \rho_{(2k-1)}^{\bar{\varepsilon}_r \varepsilon_\ell} - n \not{K} \rho_{(2k+1)}^{\bar{\varepsilon}_r +} \delta_{\varepsilon_\ell, +} + \frac{1}{\beta} \gamma^r \gamma^0 \not{K} \rho_{(2k-1)}^{\varepsilon_r -} \delta_{\varepsilon_\ell, +} \quad (\text{A.8})$$

and

$$\delta_{K_i} \rho_{(2k)}^{\varepsilon_r \varepsilon_\ell} + n\beta \gamma^v \delta_{K_i} \rho_{(2k+1)}^{\bar{\varepsilon}_r \bar{\varepsilon}_\ell} + \frac{2\omega}{\beta} \gamma^0 \delta_{K_i} \rho_{(2k-1)}^{\bar{\varepsilon}_r -} \delta_{\varepsilon_\ell, +} = K_i^{(0)} \left(\rho_{(2k)}^{\varepsilon_r \varepsilon_\ell} + n\beta \gamma^v \rho_{(2k+1)}^{\bar{\varepsilon}_r \bar{\varepsilon}_\ell} + \frac{2\omega}{\beta} \gamma^0 \rho_{(2k-1)}^{\bar{\varepsilon}_r -} \delta_{\varepsilon_\ell, +} \right) - \frac{1}{\beta} \not{K} \gamma^0 \gamma_i \rho_{(2k-1)}^{\bar{\varepsilon}_r -} \delta_{\varepsilon_\ell, +} \quad (\text{A.9})$$

$$\delta_C \rho_{(2k)}^{\varepsilon_r \varepsilon_\ell} + n\beta \gamma^v \delta_C \rho_{(2k+1)}^{\bar{\varepsilon}_r \bar{\varepsilon}_\ell} + \frac{2\omega}{\beta} \gamma^0 \delta_C \rho_{(2k-1)}^{\bar{\varepsilon}_r -} \delta_{\varepsilon_\ell, +} = C^{(0)} \left(\rho_{(2k)}^{\varepsilon_r \varepsilon_\ell} + n\beta \gamma^v \rho_{(2k+1)}^{\bar{\varepsilon}_r \bar{\varepsilon}_\ell} + \frac{2\omega}{\beta} \gamma^0 \rho_{(2k-1)}^{\bar{\varepsilon}_r -} \delta_{\varepsilon_\ell, +} \right) - \frac{in}{2} \rho_{(2k-2)}^{\varepsilon_r \varepsilon_\ell} + \frac{1}{\beta} \gamma^r \gamma^0 \rho_{(2k-2)}^{\bar{\varepsilon}_r -} \delta_{\varepsilon_\ell, +} - \frac{in^2 \beta}{2} \gamma^v \rho_{(2k-1)}^{\bar{\varepsilon}_r \bar{\varepsilon}_\ell} - \frac{in\omega}{\beta} \gamma^0 \rho_{(2k-3)}^{\bar{\varepsilon}_r -} \delta_{\varepsilon_\ell, +} + \frac{\not{K}}{\beta} \gamma^0 \not{K} \rho_{(2k-1)}^{\bar{\varepsilon}_r -} \delta_{\varepsilon_\ell, +} - \varepsilon_r n \rho_{(2k-1)}^{\varepsilon_r +}. \quad (\text{A.10})$$

Here $\bar{\varepsilon}_{r,\ell} = -\varepsilon_{r,\ell}$ and $\rho_{(m)}^{\varepsilon'_r \varepsilon'_\ell}$, $m < 0$ are defined to be zero. It is also important to note that the action of $C^{(0)}$ (the part of C acting on functions), due to the term $z\partial_z$, depends on the dimension of the fields it acts on. From (A.7) and (A.8), we see that $\rho_{(2k-1)}^{\bar{\varepsilon}_r -}$ transform into themselves

$$\not{K} \delta_{K_i} \rho_{(2k+1)}^{\bar{\varepsilon}_r -} = \not{K} K_i^{(0)} \rho_{(2k+1)}^{\bar{\varepsilon}_r -} - n\gamma_i \rho_{(2k+1)}^{\bar{\varepsilon}_r -} \quad (\text{A.11})$$

$$\not{K} \delta_C \rho_{(2k+1)}^{\bar{\varepsilon}_r -} = C^{(0)} \not{K} \rho_{(2k+1)}^{\bar{\varepsilon}_r -} - \frac{in}{2} \not{K} \rho_{(2k-1)}^{\bar{\varepsilon}_r -}, \quad (\text{A.12})$$

hence can be consistently set to zero. With that in mind, the transformations of $\rho_{(2k)}^{\varepsilon_r -}$ can be deduced from (A.7), (A.8), (A.9) and (A.10) to read

$$\not{K} \delta_{K_i} \rho_{(2k)}^{\varepsilon_r -} = \not{K} K_i^{(0)} \rho_{(2k)}^{\varepsilon_r -} - n\gamma_i \rho_{(2k)}^{\varepsilon_r -} \quad (\text{A.13})$$

$$\not{K} \delta_C \rho_{(2k)}^{\varepsilon_r -} = \not{K} C^{(0)} \rho_{(2k)}^{\varepsilon_r -} - \frac{in}{2} \not{K} \rho_{(2k-2)}^{\varepsilon_r -} + n \not{K} \rho_{(2k)}^{\varepsilon_r -}. \quad (\text{A.14})$$

Again, they only transform into themselves. Thus, we have shown that the representation can be reduced by setting $\rho_{(m)}^{\varepsilon_r -} = 0$. The transformations of the remaining fields are simplified significantly

$$\delta_{K_i} \rho_{(2k+1)}^{\varepsilon_r +} = K_i^{(0)} \rho_{(2k+1)}^{\varepsilon_r +} \quad (\text{A.15})$$

$$\delta_C \rho_{(2k+1)}^{\varepsilon_r +} = C^{(0)} \rho_{(2k+1)}^{\varepsilon_r +} - \frac{in}{2} \rho_{(2k-1)}^{\varepsilon_r +} \quad (\text{A.16})$$

$$\delta_{K_i} \rho_{(2k)}^{\varepsilon_r +} = K_i^{(0)} \rho_{(2k)}^{\varepsilon_r +} \quad (\text{A.17})$$

$$\delta_C \rho_{(2k)}^{\varepsilon_r +} = C^{(0)} \rho_{(2k)}^{\varepsilon_r +} - \frac{in}{2} \rho_{(2k-2)}^{\varepsilon_r +} - \varepsilon_r n \rho_{(2k-1)}^{\varepsilon_r +} \quad (\text{A.18})$$

Looking at (A.15)-(A.18), we notice that fields with opposite ε_r index do not mix under the transformations as well. Thus, the representation can be maximally reduced by setting one of the two $\rho_{(m)}^{\varepsilon_r +}$ to zero. The criteria is obvious: the leftover representation must include all solutions of the Dirac equation. Thus, in (A.5) it corresponds to keeping only $\rho_{(m)}^{\bar{\varepsilon}_r +}$ in the series with leading order $z^{\Delta_{\varepsilon_r}^\pm}$. This gives the off-shell spinor

$$\Psi = \sum_{\varepsilon_r} \left[z^{\Delta_{-\varepsilon_r}^-} \sum_{k=0}^{\infty} z^{2k} \left(\rho_{(2k)}^{\varepsilon_r} + \mathcal{Q} \rho_{(2k+1)}^{\varepsilon_r} \right) + z^{\Delta_{-\varepsilon_r}^+} \sum_{k=0}^{\infty} z^{2k} \left(\chi_{(2k)}^{\varepsilon_r} + \mathcal{Q} \chi_{(2k+1)}^{\varepsilon_r} \right) \right], \quad (\text{A.19})$$

where $\rho^{\varepsilon_r} = \rho^{\varepsilon_r +}$, etc.

A.2 Massless limit

In the massless limit, $\mu_+ = \mu_- = \mu$, $\Delta_-^\pm = \Delta_+^\pm = \Delta^\pm$ and the expansion (A.5) collapses into just two series. However, for each series all the analysis carried out above is still valid. The only difference is that now to include all solutions of the Dirac equation in the reduced representation, for each series we must keep both $\rho_{(m)}^{\varepsilon_r+}$ rather than just one of them. The transformation laws are the same as (A.15)-(A.18). Again, fields with opposite ε_r index do not mix under the transformations. Each series then contains two irreducible representations of the Schrödinger group, labeled by ε_r .

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